

Properties making a chaotic system a good pseudo random number generator

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We discuss the properties making a deterministic algorithm suitable to generate a pseudo random sequence of numbers: high value of Kolmogorov-Sinai entropy, high dimensionality of the parent dynamical system, and very large period of the generated sequence. We propose the multidimensional Anosov symplectic (cat) map as a pseudo random number generator. We show what chaotic features of this map are useful for generating pseudo random numbers and investigate numerically which of them survive in the discrete state version of the map. Testing and comparisons with other generators are performed.

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I. INTRODUCTION

In most scientific uses of numerical computations, e.g., Monte Carlo simulations and molecular dynamics, it is necessary to have a series of independent, identically distributed (i.i.d.) continuous random variables $x(1), x(2), \dots, x(n)$ with assigned single variable probability density function (PDF) $P(x(i))$. Of course, it is enough to have i.i.d. random variables $\{x(t)\}$ uniformly distributed in the interval $[0,1]$, since a suitable change of variable $y=g(x)$ may generate numbers $\{y(i)\}$ with any PDF $\tilde{P}(y)$.

Let us call a process producing i.i.d. variables uniformly distributed in $[0,1]$ a perfect random number generator (RNG). One can produce a perfect RNG only using non-deterministic physical phenomena, e.g., the decay of radioactive nuclei or the arrival on a detector of cosmic rays.

A more practical way is to use a computer that produces a "random-looking" sequence of numbers, by means of a recursive rule. Let us call an algorithm designed to mimic a random sequence on a computer a pseudo random number generator (PRNG). This issue is far from being trivial; in Von Neumann's words: "Anyone who considers arithmetical methods of producing random digits is, of course, in a state of sin" [1]. The two unavoidable problems are the following: (a) numerical algorithms are deterministic; (b) they deal with discrete numbers.

The limitations arising from these properties can be analyzed using the language and the tools of dynamical systems theory. In the following, we anticipate how these remarks translate in this framework and the main issues of the entropic characterization of PRNG's (these issues are discussed in detail in Sec. II).

(a) Since the algorithm is deterministic, the Kolmogorov-Sinai (KS) entropy (h_{KS}) is finite. The sequence $\{x(i)\}$ cannot be "really random," i.e., with an infinite KS entropy, because the deterministic dynamical rule constrains the outputs that are near in time and supplies us with a maxi-

imum of $\log_2(e^{h_{KS}})$ random bits per unit time. This limitation would be present also in a hypothetical computer able to work with real numbers.

(b) Since any deterministic system with a finite number of states is periodic, any sequence produced by an algorithm working with discrete numbers must be periodic, possibly after a transient: therefore, not only $h_{KS} < \infty$, but $h_{KS} = 0$. The computer-implemented system can be only pseudo-chaotic.

First, we consider point (a). After the seminal work of Lorenz [2] and Hénon [3] (to mention just two of the founders of the modern theory of chaos), it is well established that also deterministic systems may have a time evolution that appears rather "irregular" with the typical features of genuine random processes. This evidence opened a debate on the possibility of distinguishing between noisy and chaotic deterministic dynamics. Following the work of Takens [4], so-called embedding techniques have been developed to extract qualitative and quantitative information from a time series. The initial enthusiasm was because the use of the embedding method (via delayed coordinates) allows, at least in principle, the determination of quantities like dimensions, h_{KS} , and Lyapunov exponents. People believed that, after determining the KS entropy of a data sequence, one would know the true nature (deterministic or stochastic) of the law generating the series. It is now rather clear that there are several limitations in the use of this technique [5]; for instance, the number of points necessary for the phase-space reconstruction increases exponentially with the dimension of the system [6]. Thus, due to the finiteness of the data sets, it is not possible to perform an entropic analysis with an arbitrarily fine resolution, i.e., to compute the ϵ entropy $h(\epsilon)$ for very small values of ϵ . This fact severely restricts the possibility of distinguishing between signals generated by different rules, such as regular (high dimensional) systems, deterministic chaotic systems, and genuine stochastic processes. Although the above result may appear negative, it allows a pragmatic classification of the stochastic or chaotic feature of

the signal, according to the dependence of the ϵ entropy on ϵ , and this yields some freedom in modeling systems [7]. As a relevant example of a representation of a deterministic system in term of stochastic processes, we mention fully developed turbulence [8]. Turbulent systems are high dimensional deterministic chaotic systems and therefore $h(\epsilon) \simeq h_{KS}$ for $\epsilon \lesssim \epsilon_c$, where $\epsilon_c \rightarrow 0$ as the Reynolds number $Re \rightarrow \infty$, while $h(\epsilon) \sim \epsilon^{-\alpha}$ for $\epsilon \gtrsim \epsilon_c$. The fact that in certain stochastic processes $h(\epsilon) \sim \epsilon^{-\alpha}$ can be useful for modeling purposes; for example, in so-called synthetic turbulence, one introduces suitable multiaffine stochastic processes with the correct scaling properties of the fully developed turbulence.

In this paper we want to discuss the opposite strategy, i.e., mimicking noise with deterministic chaotic systems. Let us summarize the starting points of our approach to using a deterministic chaotic system as a PRNG.

(1) Since in any deterministic system $h(\epsilon) \simeq h_{KS}$ for $\epsilon \lesssim \epsilon_c$ with $(\ln \epsilon_c) \sim -h_{KS}$, one should work with a very large h_{KS} [9]. In this way the true (deterministic) nature of the PRNG becomes apparent only at a very high resolution.

(2) The outputs $\{x(t)\}$ of a perfect RNG, when observed at resolution ϵ , supply $\log_2(1/\epsilon) \propto h(\epsilon)$ random bits per iteration. In order to observe the behavior $h(\epsilon) \sim \ln(1/\epsilon)$ for $\epsilon \gtrsim \epsilon_c$ in a deterministic algorithm, it is necessary that the time correlation is very weak. We will discuss how this property may be achieved by taking as output a single variable of a high dimensional chaotic system.

It is not difficult to satisfy point (1), while request (2) is less obvious. Anosov systems [10] are natural candidates to fulfill it, having invariant stationary measure and very strong chaotic properties.

A third point has to be added, dealing with the problem (b) and its consequence (b₁).

Up to here we were considering the chaotic properties of systems with continuous phase space. Quantities like h_{KS} and the ϵ entropy have an asymptotic nature, i.e., they are related to large time behavior. However, there are situations where the system is, strictly speaking, nonchaotic ($h_{KS}=0$) but its features appear irregular to a certain extent. This property (denoted by the term pseudochaos [11–13]) is basically due to the presence of long transient effects [14].

As noted above, the use of a computer discretizes the phase space of a dynamical system, canceling (at least) its asymptotic chaotic properties. However, if the period of the realized sequence is long enough, the effects related to points (1) and (2) reasonably survive as a chaotic transient. According to this observation, we add a third request.

(3) The period of the series generated by the computer (i.e., with a state discretization of the deterministic system) must be very large.

Point (3) is really tough: as far as we know, for a generic deterministic system with \mathcal{M} discrete states, there are no general methods to determine *a priori* the length of the periodic orbits. A nice result, based on probabilistic considerations, suggests that the period $\mathcal{T} \simeq \mathcal{M}^{1/2}$ [16], although strong fluctuations are present. The use of high dimensional systems may be a natural solution also for this problem: denoting by M the number of states along each of the d dimensions, the typical period $\mathcal{T} \simeq M^{d/2}$ grows very fast on increasing M and d .

Whatever the mechanism for producing the pseudochaotic transient, the mere fact that the sequence is periodic implies that it is possible to obtain equidistributed words only up to a length $\bar{m} = O(\ln \mathcal{T})$. Thus, long time correlations among outputs of a generator cannot be detected by the standard entropic analysis. We will show that a high dimensional chaotic system provides outputs which are not correlated even looking at time delays greater than \bar{m} . In particular we will discuss the connection between correlation functions and the spectral test for random sequences [17–19] showing that the outputs of the high dimensional cat map have zero n -point correlation functions, when n is less than or equal to the dimensions of the map.

In Sec. II, we describe the entropic properties of PRNGs currently used, underlying both the mechanisms involved in PRNGs. In Sec. III the algorithms used to test the generators are described. In Sec. IV we study the properties of the multidimensional Arnold's cat map and in Sec. V we propose its discrete version as a PRNG. Section VI is devoted to conclusions and perspectives.

II. ENTROPY AND GOOD PRNGS

First of all, we briefly recall some basic notions of the ϵ entropy [13]. Consider the variable $\mathbf{x}(t) \in \mathbb{R}^d$ representing the state of a d -dimensional system, and introduce the new variable

$$\mathbf{y}^{(m)}(t) = (\mathbf{x}(t), \mathbf{x}(t+1), \dots, \mathbf{x}(t+m-1)) \in \mathbb{R}^{md}. \quad (1)$$

Of course, $\mathbf{y}^{(m)}$ corresponds to a trajectory in a time interval m . Then, the phase space is partitioned into cells of linear size ϵ in each of the d directions. Since the region where a bounded trajectory evolves contains a finite number of cells, each $\mathbf{y}^{(m)}(t)$ defined in Eq. (1) can be coded into a word of length m out of a finite alphabet:

$$\mathbf{y}^{(m)}(t) \rightarrow W_\epsilon^{(m)}(t) = (i(\epsilon, t), i(\epsilon, t+1), \dots, i(\epsilon, t+m-1)) \quad (2)$$

where $i(\epsilon, t+j)$ labels the ϵ cell containing $\mathbf{x}(t+j)$. Assuming that the sequence is stationary and ergodic, from the time evolution of $\mathbf{y}^{(m)}(t)$ the probabilities $P(\{W_\epsilon^{(m)}\})$ are computed, and one defines the block entropies of size ϵ :

$$H_m(\epsilon) = - \sum_{\{W_\epsilon^{(m)}\}} P(W_\epsilon^{(m)}) \ln[P(W_\epsilon^{(m)})]. \quad (3)$$

Finally one introduces the ϵ entropy $h(\epsilon)$:

$$h(\epsilon) = \lim_{m \rightarrow \infty} h_m(\epsilon) \quad (4)$$

where $h_m(\epsilon) = H_{m+1}(\epsilon) - H_m(\epsilon)$ represents the ϵ -block entropy growth at word length m . In a rigorous approach, all partitions into elements of size smaller than ϵ should be taken into account, and then $h(\epsilon)$ is defined as the infimum over all these partitions [20]. The KS entropy can be identified as the limit $\epsilon \rightarrow 0$:

In a deterministic chaotic system, one has $h_{KS} < \infty$, in a regular motion $h_{KS} = 0$, while for a random process with con-

tinuous states $h_{KS}=\infty$. For some stochastic processes, it is possible to give an explicit expression for $h(\epsilon)$ [21]. For instance, for a stationary Gaussian process with spectrum $S(\omega) \sim \omega^{-(1+2\alpha)}$ with $0 < \alpha < 1$ one has

$$h(\epsilon) \sim \epsilon^{-1/\alpha} \tag{6}$$

while for i.i.d. variables whose PDF is continuous in a bounded domain (e.g., independently distributed variables in $[0,1]$) one has

$$h(\epsilon) \sim \ln\left(\frac{1}{\epsilon}\right). \tag{7}$$

Of course, leaving aside the problem of the periodicity induced by the discrete nature of the states, a PRNG is good when its h_{KS} is very large, such that the uncertainty on the “next” outcome is larger and the deterministic constraints appear on scales smaller than an ϵ_c defined by $(\epsilon_c)^d \sim e^{-h_{KS}}$.

On the other hand, in data analysis, the space where the state vector \mathbf{x} exists is unknown and typically in experiments only a scalar variable $u(t)$ is measured. Therefore, in order to reconstruct the original phase space, one uses the vector

$$\mathbf{y}^{(m)}(t) = (u(t), u(t+1), \dots, u(t+m-1)) \in \mathbb{R}^m; \tag{8}$$

that is another way to coarse-grain the phase space. In this case, i.e., looking only at one variable, the maximum scale where the Kolmogorov-Sinai entropy may be revealed is given by $\epsilon_{c1} \sim e^{-h_{KS}}$, which is much smaller than ϵ_c , for large d [9]. Moreover, the single-variable word length that is necessary to consider, in order to detect h_{KS} , must be greater than d . This effect is harmful from the perspective of data analysis, but is welcome here.

We also note that in any series of finite length T , it is not possible to have a good statistics of m words at resolution ϵ if $T \lesssim e^{mh(\epsilon)}$. Therefore, for almost all the practical aims, i.e., for finite ϵ and finite size of the sequence, a chaotic PRNG with very high h_{KS} has entropic properties indistinguishable from those of a perfect RNG.

A. High entropy PRNG

Several PRNGs are indeed discrete state versions of high entropic dynamical systems. A popular example is the multiplicative congruential method:

$$z(t+1) = az(t) \pmod{M}, \tag{9}$$

where $z(t)$, a , and M are integers, with $M \gg a \gg 1$. In the following the term “map” will denote a dynamical system with continuous state space, and the term “automaton” a system with discrete state space (we always assume a discrete time). To avoid confusion, we will use the symbols $z(t), w(t), \mathbf{z}(t), \mathbf{w}(t)$ only for discrete dynamical variables (also vectorial) and $x(t), y(t), \mathbf{x}(t), \mathbf{y}(t)$ for real dynamical variables. Equation (9) corresponds to the chaotic map

$$x(t+1) = ax(t) \pmod{1} \tag{10}$$

where $x(t) = z(t)/M$. It is easy to see that the system (10) has a uniform invariant PDF in $[0,1]$ and $h_{KS} = \ln a$. Therefore, by only looking at $\epsilon \lesssim \epsilon_c \sim 1/a$, one can capture the deterministic nature of the PRNG [9].

It is worthwhile to stress that the chaotic features of the automaton are apparent only if one observes the system (9) after a coarse-graining procedure, namely, with $\epsilon \gg 1/M$. Below this level of observation, the system keeps a loose trace of the chaotic features of its continuous precursor and, at the maximal resolution, at the first time step the block entropy already assumes its maximum value $H_m(1/M) \approx \ln T$ for all $m \geq 1$ independently on the value of h_{KS} . This happens because we are observing the complete state of the (one-dimensional) automaton and suggests again that a suitable attitude is to extract partial information from high dimensional systems.

In the next subsection we show an alternative way, based on the high dimension effect, to produce random number (up to a given word length) with an automaton, even at the finest resolution achievable.

B. High dimension PRNG

It is known that nonchaotic high dimensional systems may display a long irregular regime as a transient effect [14]. In this subsection, we show that, with a proper use of a transient irregular behavior, also systems with a moderate h_{KS} may successfully generate pseudo random sequences: in these cases, one observes a transient in the block ϵ entropies $H_n(\epsilon)$, characterized by a maximal (or almost maximal) value of the slope $H_n(\epsilon)/n$, and then a crossover to a regime with the slope of the true h_{KS} of the system.

The most used class of PRNGs using this property are the so-called lagged Fibonacci generators [15], which correspond to the following map:

$$x(t) = ax(t - \tau_1) + bx(t - \tau_2) \pmod{1}, \tag{11}$$

where a and b are $O(1)$ and $\tau_1 < \tau_2$.

Notice that Eq. (11) can be written in the form

$$\mathbf{y}(t) = \mathbb{F}\mathbf{y}(t-1) \tag{12}$$

where \mathbb{F} is a $\tau_2 \times \tau_2$ matrix of the form

$$\mathbb{F} = \begin{pmatrix} 0 & \dots & a & \dots & b \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 1 & 0 \end{pmatrix} \tag{13}$$

showing explicitly that the phase space of (11) has dimension τ_2 . It is easy to prove that this system is chaotic for each value of $a, b \in \mathbb{N}$, with $a, b > 0$. The KS entropy does not depend on τ_1 and τ_2 and is of the order $\approx \ln(ab)$; this means that to obtain high values of h_{KS} we are forced to use large values of a, b ; nevertheless, the lagged Fibonacci generators are used with $a=b=1$. For these values of the parameters $e^{-h_{KS}} \approx 0.618$ and ϵ_{c1} is not small. This implies that the determinism of the system should be detectable also with a large graining. Despite these considerations, these generators work rather well: the reason is that the m words, built up by a single variable (y_1) of the τ_2 -dimensional system (12), have the maximal allowed block entropy, $H_m(\epsilon) = m \ln(1/\epsilon)$, for $m < \tau_2$, so that

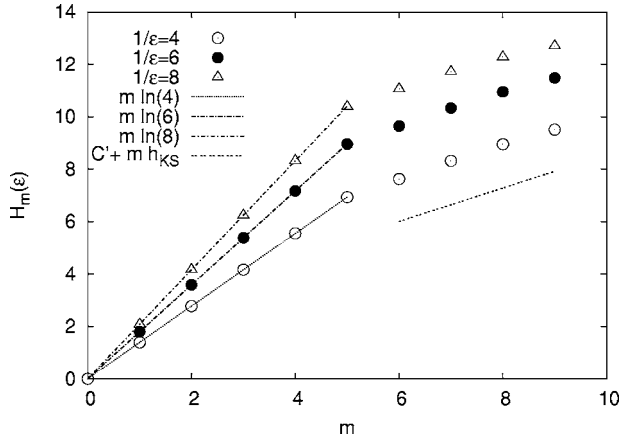


FIG. 1. The ϵ block entropy for the Fibonacci map with $\tau_1=2, \tau_2=5, a=b=1$, and different values of ϵ . The change of the slope from $\ln(1/\epsilon)$ to h_{KS} is clearly visible for $m=\tau_2=5$.

$$H_m(\epsilon) \simeq \begin{cases} m \ln\left(\frac{1}{\epsilon}\right) & \text{for } m \leq \tau_2, \\ \tau_2 \ln\left(\frac{1}{\epsilon}\right) + h_{KS}(m - \tau_2) & \text{for } m \geq \tau_2. \end{cases} \quad (14)$$

Equation (14) has the following interpretation: though the “true” h_{KS} is small, it can be computed only for a very large value of m . Indeed, by observing the one-variable m words, which corresponds to an embedding procedure, before capturing the dynamical entropy one has to realize that the system has dimension τ_2 , and this happens only for words longer than τ_2 . Figure 1 shows $H_m(\epsilon)$ for $\tau_1=2, \tau_2=5$, and different values of ϵ .

The importance of the transient behavior of H_m has been underlined by Grassberger [22] who proposed another quantity beyond the KS entropy: the “effective measure of complexity,” namely,

$$C = \sum_{m=1}^{\infty} m(h_{m-1} - h_m). \quad (15)$$

From the above definition, it follows that for large m , the block entropies grow as

$$H_m \simeq C + m h_{KS}. \quad (16)$$

For trivial processes, e.g., for Bernoulli schemes or Markov chain of order 1, $C=0$, and $h_{KS}>0$, while in a periodic sequence $h_{KS}=0$ and $C \sim \ln(\mathcal{T})$. In the case of Fibonacci map, for small ϵ ,

$$C = \tau_2 \left[\ln\left(\frac{1}{\epsilon}\right) - h_{KS} \right] \approx \tau_2 \ln\left(\frac{1}{\epsilon}\right). \quad (17)$$

For large τ_2 [usually values $O(10^2)$ are used] C is so huge that only an extremely long sequence of the order $\exp(\tau_2)$ (likely outside the capabilities of modern computers) may reveal that the “true” KS entropy is small.

Let us now discuss the behavior of the discrete Fibonacci generator

$$z(t) = az(t - \tau_1) + bz(t - \tau_2) \pmod{M}, \quad (18)$$

where $z(t) \in [0, M-1]$ and $M \gg \tau_2$. The parameters τ_1, τ_2 , and M are chosen in order to have a period as long as possible. Number-theoretical arguments [19] allow one to choose these parameters such that the period of the orbit is maximum $T=M^{\tau_2}-1$.

When the period is maximum, for $\epsilon \geq 1/M$ one has

$$H_m(\epsilon) \simeq \begin{cases} m \ln\left(\frac{1}{\epsilon}\right) & \text{for } m \leq \tau_2, \\ \tau_2 \ln\left(\frac{1}{\epsilon}\right) + h_{KS}(m - \tau_2) & \text{for } \tau_2 \leq m \leq m^*, \\ \tau_2 \ln(M) & \text{for } m \geq m^*, \end{cases} \quad (19)$$

where

$$m^* = \frac{\tau_2}{h_{KS}} \left[\ln\left(\frac{1}{\epsilon}\right) - \ln M + h_{KS} \right]. \quad (20)$$

When $\epsilon=1/M$ we have $m^*=\tau_2$, the second regime in Eq. (19) disappears, and the block entropy behavior is independent of h_{KS} . Still, as for the continuous case, if τ_2 is large one observes only the pseudochaotic transient

$$H_m(\epsilon) \approx m \ln\left(\frac{1}{\epsilon}\right). \quad (21)$$

Summarizing, systems with high values of h_{KS} or with high dimension produce sequences having entropic properties rather close to those of a perfect RNG, for two different reasons. In the first case, the large h_{KS} allows one to use a small ϵ (but large enough to achieve a proper coarse graining): in such a way, the ϵ entropy coincides with that of a perfect RNG. In the second case, the high dimensionality of the system prevents the entropic analysis from revealing the asymptotic value of h_{KS} before the end of a long transient behavior mimicking a complete random system.

In conclusion, a deterministic PRNG has good entropic properties for long (but finite) sequences if h_{KS} or C are large. In Sec. IV we will propose a multidimensional cat map as a PRNG having both these properties.

III. TESTS FOR PRNGS

Several techniques have been developed in order to test “how random” is a given sequence of numbers. These algorithms are available in easy-to-use software packages collecting dozens of different tests, like, for example, the DIEHARD [23] and the NIST [24] batteries. Many of them compute the frequency of some words $f(z(j), z(j+1), \dots, z(j+n))$ made up of n consecutive outputs of the generator and compare it to the theoretical probability in the random case. Examples are the frequency and block-frequency tests [computing $f(z(j))$, i.e., $m=1$], the poker test looking for words with $m=5$ corresponding to the poker hands (e.g., full house 00011; four a kind 00001), and template tests checking the occurrences of some ($\approx 10^2$) words with length $m=10-12$.

It is worth stressing that all these benchmarks are automatically passed if the block ϵ entropy, is maximal for words of length m , namely,

$$H_m(\epsilon) = m \ln(M) \tag{22}$$

with $\epsilon = 1/M$ where M is the number of symbols produced by the generator. When H_m is maximal, it is impossible to distinguish the output from a truly random sequence by looking only at m consecutive symbols. The reason for introducing many tests, instead of looking only at the entropy, is that a little departure from a constant word frequency implies a correction in the entropy that is only quadratic in the deviation [this is a simple consequence of the fact that the value of the entropy in Eq. (22) is the maximum achievable]. Thus, it is really difficult to numerically observe imbalances in the frequency of some specific words by studying only the block entropies.

A. The spectral test

The entropic analysis is a very powerful tool from a theoretical point of view but it presents a major limitation: in a phase space with a finite number of states the block entropy cannot be larger than $\ln \mathcal{T}$, where \mathcal{T} is the period of the orbit. This fact essentially fixes an upper bound on the number of possible equidistributed words and on their length. Even with the longest periods available in currently used algorithms, the bound on the length is of order 10^2 – 10^3 . On the other hand, computer simulations often necessitate a large amount of random numbers, and long-term relationships among these numbers can be sources of hard-to-discover biases [26–28]. Therefore, less severe tests than the entropic one are needed. One can ask that the correlations among different outputs (or, more generally, among different *functions* of the outputs) vanish even when the outputs are at a time distance greater than the scale where Eq. (22) ensures the equidistribution of the words. On this time scale, numbers should *appear* random, as far as one is interested in statistical observables, even if, looking at the whole sequence, later numbers are completely determined by previous ones. Indeed, according to the entropic analysis, the knowledge of approximately $\ln \mathcal{T}$ consecutive outputs permits one to determine exactly the discrete starting condition of the system and consequently to predict the whole sequence, removing any randomness from it.

The main tool to analyze the property of correlation functions is the *spectral test*. We start by defining the frequency $f(z(t_1), \dots, z(t_n))$ of the word $(z(t_1), \dots, z(t_n))$ as we did for the Kolmogorov entropy, where now the t_i 's are generic times and the $z(t_i)$'s are not in general consecutive outputs. The spectral test is the multidimensional Fourier transform of $f(z(t_1), \dots, z(t_n))$

$$\hat{f}(s_1, s_2, \dots, s_n) = \left\langle \exp\left(\frac{2\pi i}{M} \sum_j s_j z(t_j)\right) \right\rangle, \tag{23}$$

where $s_j \in [0, M-1]$, M is the number of the discrete states, and $\langle \dots \rangle$ denotes the average over the trajectory. For true random numbers

$$\hat{f}(s_1, s_2, \dots, s_n) = \delta_{s_1,0} \delta_{s_2,0} \dots \delta_{s_n,0} \tag{24}$$

for any choice of n and of the time lags t_i s. Values of the function $\hat{f}(s_1, s_2, \dots, s_n)$ significantly different from 0 denote wave vectors of probability density fluctuations in the lattice $z(t_1), z(t_2), \dots, z(t_n)$. These fluctuations can be safely neglected only when their characteristic length scale (which we assume to be the inverse of the modulus of the wave vector) is much smaller than the maximum precision one is interested in. Many generators (i.e., the linear congruential class of generators) produce numbers that “fall mainly in planes” [17–19], and the presence of these planes is detected by the spectral test.

The importance of the spectral test is related to the fact that analytical or semianalytical methods [19,29] allow for a fast calculation for simple systems. Furthermore, since any L^2 function can be written as a Fourier series, condition (24) implies the vanishing of any correlation of up to n functions of time-delayed variables:

$$\begin{aligned} &\langle g_1(z(t_1))g_2(z(t_2)) \dots g_n(z(t_n)) \rangle \\ &= \langle g_1(z(t_1)) \rangle \langle g_2(z(t_2)) \rangle \dots \langle g_n(z(t_n)) \rangle \end{aligned} \tag{25}$$

for every $g_i \in L^2$.

IV. THE CAT MAP AS A RANDOM NUMBER GENERATOR

Recently some authors [30] have proposed the use of the Arnol'd cat map as a PRNG. We will briefly recall the properties of this map and then propose a multidimensional version with N coupled maps, showing that this generalization gives rise to very good statistical properties. In particular, it has both the properties analyzed in Sec. II giving maximal ϵ entropy, namely, it possesses a high value of h_{KS} and it is a high dimensional system. We will see that this system has also very good properties from the point of view of correlation functions.

The two-dimensional Arnol'd cat map [31] is a symplectic automorphism on a torus satisfying the properties of Anosov systems [32], namely, it is everywhere hyperbolic and has a positive Kolmogorov entropy. The map reads

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & a \\ b & 1+ab \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \pmod{1}, \tag{26}$$

where $a, b \in \mathbb{N}$. The standard example given by Arnol'd is obtained with $a=b=1$.

The multidimensional generalization can be written in the following way:

$$\begin{pmatrix} \mathbf{x}' \\ \mathbf{y}' \end{pmatrix} = \mathbb{M} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \pmod{1}, \tag{27}$$

with

$$\mathbb{M} = \begin{pmatrix} \mathbb{I} & \mathbb{A} \\ \mathbb{B} & \mathbb{I} + \mathbb{B}\mathbb{A} \end{pmatrix} \tag{28}$$

where \mathbb{M} is a $2N \times 2N$ matrix, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$, \mathbb{I} is the $N \times N$ identity matrix, and \mathbb{A}, \mathbb{B} are symmetric $N \times N$ matrices with

natural entries in order to obtain a continuous mapping. It is easy to see that the evolution law given by Eq. (28) is symplectic, indeed one can write Eqs. (27) and (28) as a canonical transformation

$$\mathbf{x} = \frac{\partial S(\mathbf{x}', \mathbf{y})}{\partial \mathbf{y}}, \quad \mathbf{y}' = \frac{\partial S(\mathbf{x}', \mathbf{y})}{\partial \mathbf{x}'} \quad (29)$$

where the generating function is given by

$$S(\mathbf{x}', \mathbf{y}) = \sum_{j=1}^N x'_j y_j - \frac{1}{2} \sum_{j,k=1}^N (y_j A_{jk} y_k + x'_j B_{jk} x'_k). \quad (30)$$

It can be shown that when $\text{Tr}(\mathbf{M}) > 2N$ the map (27) is an Anosov system with uniform invariant measure. The output of our generator will be the first component of the vector \mathbf{x} . The condition $N > 1$ raises the Kolmogorov entropy, cancels the correlation, and increases the length of the periodic orbits (in the discretized case). We will describe in detail these three aspects in the following.

First of all, according to the Pesin identity, the Kolmogorov entropy of this system is equal to the sum of the positive Lyapunov exponents:

$$h_{KS} = \sum_{\lambda_i > 0} \lambda_i. \quad (31)$$

A d -dimensional hyperbolic symplectic system possesses exactly $d/2$ positive Lyapunov exponents; if they are of the same order of magnitude, the Kolmogorov entropy grows proportionally to the number of dimensions. Notice that the Kolmogorov entropy may also be raised by simply taking the matrix \mathbf{A}, \mathbf{B} with very large entries. This method, however, produces only an increase in the entropy which is logarithmic in the size of the entries. Of course for the system (27) and (28) the λ_i are easily obtained from the eigenvalues α_i of \mathbf{M} : $\lambda_i = \ln|\alpha_i|$.

For the two-dimensional (2D) cat map we compute an approximate value of the ϵ entropy, obtained as $H_4(\epsilon) - H_3(\epsilon)$ varying ϵ and the parameters a, b . In order to highlight the practical limitations of the PRNG we study the discrete version of Eq. (26) with $M=2^{20}$ possible values of x and y (see the next section for details).

As one can observe in Fig. 2, both the standard problems of PRNGs appear. Indeed the figure shows that on decreasing the value of ϵ we observe a ‘‘plateau’’ around the value of h_{KS} . At lower values of ϵ , there is an abrupt decrease due to the periodic nature of the map. Nevertheless it seems that if we use the map as a generator of a number of symbols $\approx 1/\epsilon$ with $\epsilon > \epsilon_c$ we are, with a good approximation, near the value corresponding to a theoretical RNG, given by $h(\epsilon) = -\ln(\epsilon)$. Let us note that, when h_{KS} is large enough (curves for $a=5, b=7, h_{KS} \approx 3.61$ and $a=11, b=17, h_{KS} \approx 5.24$), because of the limited number of allowed states, one does not observe the plateau $h(\epsilon) \sim h_{KS}$.

Let us study the properties of the time correlation of the outputs. The following result holds. Let \mathbf{e}_1 be the $2N$ -dimensional vector $(1, 0, 0, \dots)$. If the vectors $(\mathbf{M}^T)^{t_1} \mathbf{e}_1, (\mathbf{M}^T)^{t_2} \mathbf{e}_1, \dots, (\mathbf{M}^T)^{t_{2N}} \mathbf{e}_1$ are linearly independent, then one has

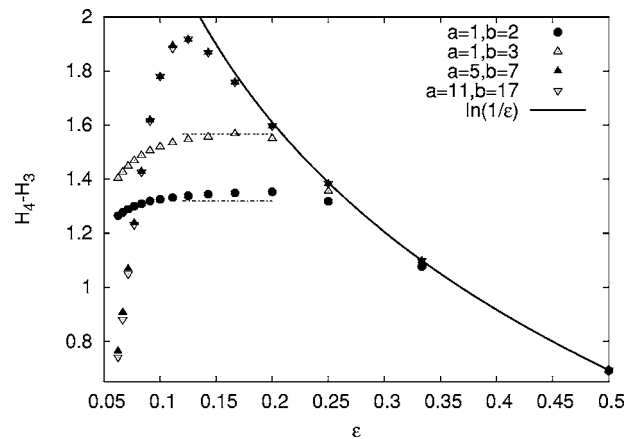


FIG. 2. $H_4(\epsilon) - H_3(\epsilon)$ for the 2D cat map of Eq. (26) with $M = 2^{20}$ and different values of a, b as a function of ϵ . The horizontal lines indicate the h_{KS} values.

$$\left\langle \exp \left(2\pi i \sum_{j=1}^{2N} s_j x_1(t_j) \right) \right\rangle = \delta_{s_1, 0} \delta_{s_2, 0} \cdots \delta_{s_{2N}, 0}. \quad (32)$$

Furthermore, the independence of the vectors is ensured for any choice of the time delays t_i s if the matrix \mathbf{M} has real, positive, and nondegenerate eigenvalues and the vector \mathbf{e}_1 has a nonzero component on all the eigenvectors. For the proof see the Appendix.

The practical meaning of this result is the following: we observe only the variable x_1 , keeping the remaining $2N - 1$ variables hidden, and we study its correlation functions. In this way correlation functions involving up to $2N$ different times vanish, i.e., Eq. (25) holds for $n \leq 2N$, because the contributions due to different values of the hidden variables cancel out in the averaging.

The result of Eq. (32) can be taken as one of the strongest characterizations of a finite random sequence: as we said, word equidistribution can hold only up to a value of $\bar{n} \approx \ln T$ where T is the length of the sequence. Indeed, some authors [25] define as a random sequence of length T one containing all the possible words up to length \bar{n} . On the other hand, Eq. (32) is a generalization of that condition: for consecutive time delays $t_j = j$ the two properties are equivalent, while for generic values of the t_j it ensures long-range independence of the outputs, without asking for an exponential number of equiprobable words in the sequence.

The validity of the property (32) in the discrete case will be the subject of careful analysis in the following section. Here, we just point out that, even in the continuous case, this property is not shared by some of the dynamical systems used for generating random numbers. For example, let us recall the Fibonacci map

$$x(t) = ax(t - \tau_1) + bx(t - \tau_2) \pmod{1} \quad (33)$$

with $a, b \in \mathbb{N}$ with $a, b > 0$ and $\tau_2 > \tau_1$. It is straightforward to show that the correlation function

$$\langle \exp\{2\pi i[s_1x(t) + s_2x(t - \tau_1) + s_3x(t - \tau_2)]\} \rangle \quad (34)$$

is not equal to zero for the vector $\mathbf{s}=(s_1, s_2, s_3)=(1, -a, -b)$. Thus, even if the dimension of the phase space τ_2 may be very high, the three-point correlation function is sufficient to unveil the deterministic nature of the system.

Therefore, when the dimension of the cat map, $2N$, is equal to the dimension of the Fibonacci generator τ_2 , both the systems guarantee that words of length $2N$ are equidistributed. However, the main advantage of the multidimensional cat map is that also the words made up of $2N$ nonconsecutive symbols are equidistributed. This property does not hold for Fibonacci generators and in some case this can lead to serious problems. A famous example is the ‘‘Ferrenberg affair’’ [26]: persistence in binary Fibonacci generators gives misleading results in Monte Carlo simulations. This problem is well analyzed in the framework of information theory in [27].

In the next section we numerically study the discrete version of the multidimensional cat map and we check whether the good statistical properties of the system survive in this case.

V. NUMERICAL ANALYSIS AND TEST OF THE MULTIDIMENSIONAL CAT AUTOMATON

A digital computer cannot handle real numbers. What a computer really calculates is a finite-digit dynamics that can be represented as a dynamics on integers. We will consider in the following the multidimensional cat automaton, namely

$$\begin{pmatrix} \mathbf{z}' \\ \mathbf{w}' \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{A} \\ \mathbf{B} & \mathbf{I} + \mathbf{BA} \end{pmatrix} \begin{pmatrix} \mathbf{z} \\ \mathbf{w} \end{pmatrix} \pmod{M}, \quad (35)$$

where, as usual, \mathbf{A}, \mathbf{B} have natural entries $z_i, w_i \in [0, 1, \dots, M-1]$.

The first problem in passing from the continuous to the discrete case is that the system has a finite number of state M^{2N} and consequently it must be periodic and it is no longer truly chaotic. The optimal condition is that there is only one orbit covering all the states but the origin $\mathbf{z}=\mathbf{w}=\mathbf{0}$ (which is a fixed point) thus obtaining a period $T=M^{2N}-1$. A peculiar feature of cat maps is that periodic orbits of the continuum system have rational coordinates [33]. Consequently, the orbits of the discretized version of the map are completely equivalent to periodic orbits of the continuous system with coordinates $z_i/M, w_i/M$. As a corollary, the map being invertible, periodic orbits do not have any transient: every state is recurrent.

Unfortunately the cat map has typically many orbits, and the great majority of them are of the same length. A theoretical analysis of these orbits has been made [33] in the two-dimensional case. For generic dynamical system, arguments based on random maps suggest that the average length of the orbits T should be roughly the square root of the total number of states [3]: in our case $T \approx M^N$. This scaling have been numerically observed in typical chaotic dynamical systems [34]. In Fig. 3, we show the length T of the orbits as a function of M and N . Despite the large fluctuations, one retrieves the expected qualitative scaling and, more impor-

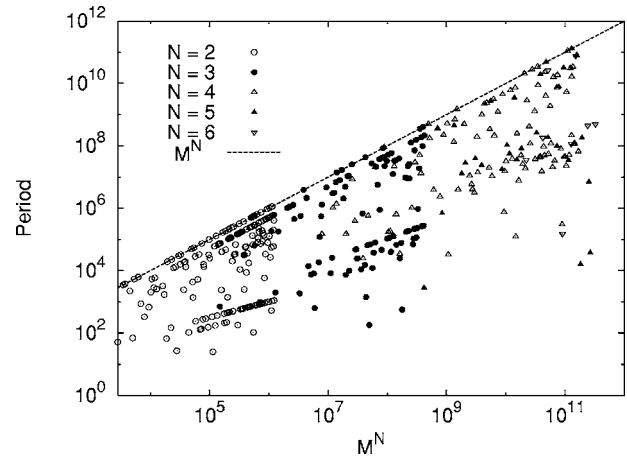


FIG. 3. The period of the orbits for the multidimensional cat map, with different values of N , as a function of M^N (only the prime values of M are considered). The straight lines refer to the probabilistic argument $T \approx M^N$.

tantly, the periods seem to be independent of the initial condition; this suggests that several symmetry operations exist mapping one orbit into another one of the same length. Since the choice of M is critical in determining the length of the orbits, we restrict ourselves to prime number values of M , in order to avoid the presence of trivial invariant sublattices generated by the divisors of M . Notice also that data are plotted as a function of M^N : the lengths show the correct exponential growth with N , at least in a statistical sense. We stress that in Fig. 3 we show the result for M not very large ($<10^3$); from the observed behavior one can say that for $M \sim 10^9$ the period should be extremely large.

Unfortunately we have no theoretical control over the period, and wild fluctuations are present when M varies; therefore it is better to choose a value of $M, N, \mathbf{A}, \mathbf{B}$ and directly check the value of T or a lower bound. In the following we will consider the choice $N=3, M=1001400791$, and

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 5 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 7 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 9 \end{pmatrix}. \quad (36)$$

With these parameters, the hypothesis leading to Eq. (32) holds; furthermore we numerically obtained $T > 7 \times 10^{12}$, which is a satisfying lower bound for typical simulations.

The very encouraging result we obtained for the correlation functions in the continuous case is the main reason for the use of the multidimensional cat automaton as a PRNG, but we have to check whether the property proven in the previous section holds also in the discrete case. The choice of (36) satisfies the hypothesis of the theorem, namely, the eigenvalues are all strictly positive and nondegenerate and the vector $(1, 0, 0, 0, 0, 0)$ has a nonzero component along all eigenvectors.

From a general point of view there are two main technical caveats when passing from continuous to discrete systems concerning the theoretical spectral test. In our case, since the orbits do not cover all the space, it is *a priori* impossible to

average on the uniform distribution as discussed in the Appendix for the continuous state case. Even if we suppose that the orbit is sufficiently homogeneous and the value of the average in (32) is close to that given by (A2) (see the Appendix), condition (A4) becomes in the discrete case

$$\sum_{j=1}^{2N} s_j m_{1k}^{(t_j)} = 0 \pmod{M} \quad \forall k = 1, \dots, 2N, \quad (37)$$

since it is sufficient to consider vectors $(s_1, s_2, \dots, s_{2N})$ in the first Brillouin zone.

These two reasons prevent us from performing the spectral test using the nice theoretical arguments used for other kinds of generators [19,35] and force us to use numerical simulation. This constitutes a hard computational task and we perform the test for low values of M and study products of the form

$$\hat{f}(s_1, s_2) = \left\langle \exp\left(\frac{2\pi i}{M} s_1 z_1(t_1)\right) \exp\left(\frac{2\pi i}{M} s_2 z_1(t_2)\right) \right\rangle. \quad (38)$$

We use $M=1031$ and $N=3$ obtaining a period $\mathcal{T}=274,243,921$ and letting $s_1, s_2 \in [0, M-1]$. Using a fast Fourier transform numerical algorithm we check up to time delays $t_2 - t_1 \leq 250$ that for all values but $s_1=s_2=0$, $|\hat{f}(s_1, s_2)| < 10^{-5}$. With a lower number of states, $M=127$, $\mathcal{T}=1,016,188$, we compute also the three-point spectral test, obtaining always values compatible with the inverse of the square root of the period \mathcal{T} . This suggests that the periodic orbits look like a finite statistical sample of the continuum equilibrium distribution, as long as one studies only few-point correlation functions. An important remark is that a solution $(s_1^*, s_2^*, \dots, s_{2N}^*)$ of the diophantine Eq. (37) implies that $\hat{f}(s_1^*, s_2^*, \dots, s_{2N}^*) = 1$ independently of the stationary distribution and, consequently, of the period \mathcal{T} . This means that the low values observed in the numerical spectral test exclude the possibility of a solution in Eq. (37).

In order to look for any other possible bias, we also apply the NIST battery to test our multidimensional cat automaton, with the parameters of Eq. (36), for generating 10^3 binary strings of 0's and 1's of length 10^6 ; all tests performed with the recommended parameters have been passed.

VI. CONCLUSIONS

In this paper we show how, using properties of high dimensional deterministic chaotic systems, it is possible to generate a good approximation of a random sequence, in spite of unavoidable constraints of deterministic algorithms running on digital computers.

Summarizing, we have two possible mechanisms to obtain good PRNGs using deterministic systems: very high KS entropy, and “transient chaos” with a large finite-time ϵ entropy, due to the high dimensionality of the system. We propose the multidimensional cat map as a PRNG having both these properties. Another important example of a system with both the properties is the one proposed by Knuth [36]: one iterates the Fibonacci generator (11) with $M=2^{31}-1$,

$\tau_1=37$ and $\tau_2=100$ (with this choice the period is extremely large); then the output sequence is obtained taking the variable in Eq. (11) every T steps ($T=1009$ or 2009). In such a way, for words of size up to τ_2 (i.e., extremely huge), the ϵ entropy is practically $\approx \ln(1/\epsilon)$, i.e., as for a perfect RNG. Moreover, even if the simple Fibonacci generator fails the three-point spectral test, it is harder to find a non-null vector in the spectral test of Knuth's generator, because of the fact that T , τ_1 , and τ_2 are relatively prime numbers. Nevertheless, it does not seem that a general result like that of Eq. (32) may be easily extended to this PRNG.

We suggest that the multidimensional cat map is suitable for generating a random number sequence. The main advantage if compared with other generators is the factorization of all the n -time, correlation functions, with $n < 2N$, due to the high dimensionality of the system and the presence of hidden variables. This result is rigorously true (also in the case $n=2N$) in the continuous system; numerical checks show that this property survives in the discrete case. Moreover, this map has a large value of the KS entropy giving good entropic properties at nonzero, but small, ϵ .

A disadvantage of this method is that we cannot predict analytically the period given the parameters or, equivalently, write a condition on the parameters in order to obtain the maximum period. However, probabilistic arguments [16], confirmed by a numerical check, show that the period increases exponentially with N ; therefore with a proper choice of the parameters we achieve extremely large periods. An analytical criterion to predict the length of the period could pave the way to the application of multidimensional cat maps as high quality PRNGs.

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APPENDIX

Consider the system of Eqs. (27) and (28). In this appendix we give the proof of the following proposition.

Let \mathbf{e}_1 be the $2N$ -dimensional vector $(1, 0, 0, \dots)$. If the vectors $(\mathbb{M}^T)^{t_1} \mathbf{e}_1, (\mathbb{M}^T)^{t_2} \mathbf{e}_1, \dots, (\mathbb{M}^T)^{t_{2N}} \mathbf{e}_1$ are linearly independent, then one has

$$\left\langle \exp\left(2\pi i \sum_{j=1}^{2N} s_j x_1(t_j)\right) \right\rangle = \delta_{s_1,0} \delta_{s_2,0} \cdots \delta_{s_{2N},0}. \quad (\text{A1})$$

Furthermore, the independence of the vectors is ensured for any choice of the time delays t_j if the matrix \mathbb{M} has real, positive, and nondegenerate eigenvalues and the vector \mathbf{e}_1 has a nonzero component on all the eigenvectors.

Proof. Since the system under study is ergodic and its invariant measure is uniform, we can write the average in Eq. (32) as

$$\int dx_1 \cdots dx_{2N} \exp\left(2\pi i \sum_{j=1}^{2N} s_j x_1(t_j)\right). \quad (\text{A2})$$

Here, with an abuse of notation, we call the components of both the \mathbf{x} and the \mathbf{y} vectors x_j , i.e., $x_{j+N} \equiv y_j$, $j=1, \dots, N$. Let us also call the elements of the matrix \mathbb{M}^t $m_{jk}^{(t)}$. We can rewrite the previous expression in the following way:

$$\int dx_1 \cdots dx_{2N} \exp\left(2\pi i \sum_{j,k=1}^{2N} s_j m_{jk}^{(t)} x_k\right). \quad (\text{A3})$$

Notice that we do not take care about the modulus since the s_j are integers and the complex exponential is periodic. When integrating over the x_i 's, the result is zero for every value of the s_i 's, excluding the values that are solutions of the linear system

$$\sum_{j=1}^{2N} s_j m_{jk}^{(t)} = 0 \quad \forall k=1, \dots, 2N \quad (\text{A4})$$

that yield 1 as a result of Eq. (A3). Since $s_j=0 \forall j$ is a trivial solution for the linear system, it is sufficient to show that this solution is unique to demonstrate Eq. (32). In particular, by Cramer's rule, it is sufficient to show that $\det G \neq 0$, where G is the matrix of coefficients of the linear system (A4), namely,

$$g_{kj} = m_{jk}^{(t)}. \quad (\text{A5})$$

Notice that the columns of the matrix G are constituted by the components of the vectors $(\mathbb{M}^T)^t \mathbf{e}_1$; this means that the condition of having $\det G \neq 0$ is equivalent to requiring that the vectors $(\mathbb{M}^T)^t \mathbf{e}_1$ are linearly independent. This completes the first part of the proof.

Now, we show how Eq. (A4) has always a unique solution when the matrix \mathbb{M} has positive and nondegenerate eigenvalues λ_k , and the vector \mathbf{e}_1 has nonzero components along all the eigenvectors. In this case, we rewrite the matrix G in the eigenvector basis, obtaining

$$\det G = c_1 c_2 \cdots c_{2N} \det \begin{pmatrix} \lambda_1^{t_1} & \lambda_1^{t_2} & \dots & \lambda_1^{t_{2N}} \\ \lambda_2^{t_1} & \lambda_2^{t_2} & \dots & \lambda_2^{t_{2N}} \\ \dots & \dots & \dots & \dots \\ \lambda_{2N}^{t_1} & \lambda_{2N}^{t_2} & \dots & \lambda_{2N}^{t_{2N}} \end{pmatrix} \quad (\text{A6})$$

where $c_k \neq 0$ (for hypothesis) is the component of \mathbf{e}_1 along the k th eigenvector. Notice that the c_k 's are real: since the eigenvalues are real by hypothesis, also the eigenvectors have real components in the natural basis of \mathbb{R}^{2N} . The proof is a reduction *ad absurdum*. Let us suppose that $\det G=0$; this implies that there exists a linear combination of the columns satisfying

$$\sum_j b_j \lambda_k^{t_j} = 0 \quad \forall k=1, \dots, 2N. \quad (\text{A7})$$

Eq. (A7) implies that the polynomial

$$P(z) \equiv \sum_j b_j z^{t_j} \quad (\text{A8})$$

has $2N$ distinct real positive roots since, by hypothesis, all eigenvalues are positive and nondegenerate. Then, by Descartes' sign rule, it must have at least $2N$ sign changes in the coefficients, but this is impossible, since $P(z)$ has just $2N$ terms different from 0. Thus, $\det G$ is necessarily different from zero for any possible choice of the time delays; this completes the proof.

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